Zero-Mass Leptonic Decays of W Bosons*

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The leptonic decays of the intermediate vector bosons with radiative corrections is studied in the limit where the mass of the charged lepton tends to zero. By means of a simple rule of calculation, we show the independence of the problem on the boson spin. Thus the ratio of the decay rates $(W \to e\nu)/(W \to \mu\nu')$ is obtainable from the corresponding ratio for a spin-zero boson interacting with a weak scalar current. While the radiative correction to the ratio of total decay rates is zero in this approximation, the result is also given for the nonzero correction in the case where discrimination against high-bremsstrahlung photon energies is applied. The spin independence is shown to be true for all boson spins (to the lowest order in α).

WHILE a study of the radiative corrections to the leptonic decays of the W bosons of weak interaction theory has no immediate experimental relevance, it does have some theoretical pertinence. First, the limit as the charged-lepton (physical) mass tends to zero is of interest as a check on a theorem due to Kinoshita and Sirlin,[1] which would allow no zero-mass divergence in the total rate of leptonic decays, in any order of α. Furthermore, with the new formulation of the vector-boson electrodynamics,[2,3] Lee has already computed that part of the radiative corrections to W-leptonic decays which is of order $(m_l^2/m_W^2)\alpha \ln\alpha$ by summing an infinite series. Since the infinite sum in addition contributes terms only of order $\alpha(m_l^2/m_W^2)\ln m_l$, any $\alpha \ln m_l$ contributions, if they exist, can be obtained from the lowest order graphs.

In this paper we develop a method of calculation which reproduces, in the limit where the charged-lepton mass tends to zero, the leading divergent terms resulting from the lowest order radiative-correction integrals. Using this procedure, we are able to show the irrelevance of the spin of the decaying boson to our problem. More specifically, we show that even for individual Feynman graphs the problem of the vector boson is equivalent to the corresponding one for a spin-zero boson interacting weakly with a scalar current, so far as the leading zero-mass divergent terms are concerned. Therefore, we can carry over the result from the spin-zero case, which has already been worked out by Kinoshita.[4] The radiative correction to the ratio of total rates of W decay into the $(e\nu)$ mode relative to the $(\mu\nu')$ mode is zero in this limit, in accordance with the theorem mentioned above. We have included the result for the case where discrimination against bremsstrahlung high-energy photons is applied. Being nonzero in our approximation, this result is numerically more definite than the ratio of total rates. For a boson mass around a kaon mass, the radiative correction may be as large as $\sim -15\%$.

In an effort to understand the effect of the spin on the problem, we have extended our arguments to include arbitrary boson spins. We can establish, for a given Lagrangian of a meson with some arbitrary spin, a correspondence, as $m_l \to 0$, with a suitably chosen decay Lagrangian for a spin-zero meson. This we show only to the lowest order in α.

I. $W^+ \to l^+ + \nu$

We study in this section the pure decay matrix element. In order to be definite, we refer directly to the electron mode. For easy reference, we first show briefly the pure decay dynamics.

The matrix element is given, in general, by (see Fig. 1)

$$M = (2\pi)^4 \delta^{(4)}(k-p-q)\mathfrak{M}_\rho \eta_\rho/(2k_0)^{1/2}, \quad (1)$$

where η_ρ is the vector-boson polarization, and $\mathfrak{M}_\rho$ is given by

$$\mathfrak{M}_\rho = \bar{u}(q)(1-\gamma_5)[a\gamma_\rho + ibk_\rho + icp_\rho]v(p). \quad (2)$$

a, b, c are, in general, functions of k^2, $k\cdot p$ in the region of physical decay. They are all real. b, for physical decay, does not contribute, since $k\cdot\eta$ is then zero. c gives rise to a positron that in the limit $m(=m_e) \to 0$ has a negative helicity, which is opposite to the corresponding one arising from the term a. Thus, we should expect that in a theory where the bare fundamental coupling is characterized by $a = g_0$, $b = c = 0$, the wrong helicity term, c, which is induced by electromagnetic corrections, must vanish in the limit $m \to 0$.

The general expression for the rate of $W^+ \to e^+\nu$

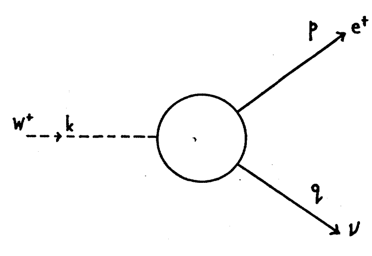

FIG. 1. $W^+ \to e^+\nu$ decay.

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[1] T. Kinoshita and A. Sirlin, Phys. Rev. **113**, 1652 (1959), and T. Kinoshita, J. Math. Phys. **3**, 650 (1962). As it turns out, this theorem is a special case of a more general theorem due to M. Nauenberg and T. D. Lee (private communication). We thank Professor Lee for a lecture on this.

[2] T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962).

[3] T. D. Lee, Phys. Rev. **128**, 899 (1962).

[4] T. Kinoshita, Phys. Rev. Letters **2**, 477 (1959).

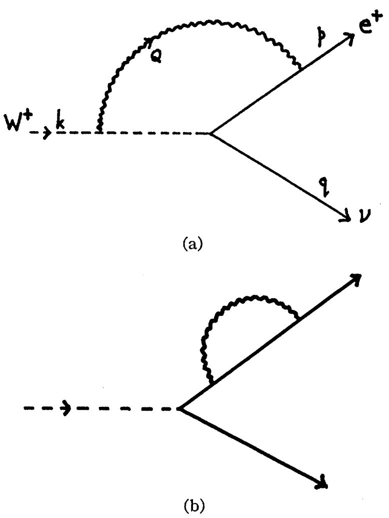

FIG. 2. Perturbation diagrams for lowest order virtual photon correction to the $W^+ \to e^+\nu$ process.

decay in W^+ rest frame is for a given helicity s of e^+,

$$R(W^+ \to e^+\nu) = \frac{\mu}{6\pi}\left(\frac{2v}{1+v}\right)^2 \left\{ a^2\frac{(1+s)}{2} + \frac{(1-s)}{2} \times\left[\frac{m^2}{2\mu^2}a^2 + \frac{\mu^2}{2}\frac{v^2}{(1+v)^2}c^2 - m\frac{v}{1+v}ac\right]\right\}, \quad (3)$$

where s = helicity of e^+, $\mu = m_{W}$, $m = m_e$, and v = velocity of e^+ in W^+ rest frame $= (\mu^2 - m^2)/(\mu^2+m^2)$. Equation (3) clearly shows the difficulty which c, if nonvanishing with $m \to 0$, leads to.

For the bare matrix element, the ratio of the total rate of decay into the $(\mu\nu')$ mode is given by

$$R_0 = \frac{R(W^+ \to e^+\nu)}{R(W^+ \to \mu^+\nu')} = \frac{v_e^2(1+v_\mu)^3(3+v_e)}{v_\mu^2(1+v_e)^3(3+v_\mu)}. \quad (4)$$

We turn now to an actual investigation of the inclusion of the lowest order virtual photon effects. As is well known, the new formulation of the electrodynamics of vector mesons leads to a result of the form[3] ($m \to 0$).

$$\alpha = Zg_0 - g_0 \cdot \frac{\alpha}{16\pi}\ln(\alpha K^2)\cdot[1+\tfrac{5}{6}K] + O(\alpha) \qquad (b,\, c \to 0 \quad \text{as } m \to 0), \quad (5)$$

where Z is some constant remaining after vector meson renormalization. The calculation was done to order $\alpha \log\alpha$. Clearly (5) is inadequate for the present problem, since the $\alpha \log\alpha$ term (and, indeed, Z also) simply cancels in the ratio of rates. In other words, our attention should now center on the $O(\alpha)$ term.

To be sure, part of the $O(\alpha)$ term involves an infinite sum. It is easy to see, however, that this infinite sum does not diverge in the limit $m \to 0$, and therefore, does not contribute to the log m terms. For the infinite sum is associated with $\xi \to 0$ divergent terms which come from the ultraviolet region of integration. In such a high-energy region m is clearly immaterial, and so we should expect that no additional divergence arises from setting $m = 0$. This then disposes of the infinite sum, which we would not have been able to handle in any case.

Therefore, we look for log$m$ terms from the rest of the integrals. In terms of Feynman graphs, there are only two diagrams which can contribute log$m$ terms (see Fig. 2). The second of these, Fig. 2(b), is the electron renormalization graph, which is sufficiently well known. It contributes to the ratio of rates a term

$$\Delta R[\text{Fig. } 2(\text{b})] = R_0\left(-\frac{3\alpha}{2\pi}\ln\frac{m_\mu}{m_e}\right). \quad (6)$$

Evidently, this term cannot depend on the spin of the decaying meson.

We turn now to a detailed consideration of Fig. 2(a). Here, spin seemingly complicates the algebra somewhat and log$m$ terms become consequently harder to enumerate. Obviously a simple and concise method of calculation is needed. We describe here one such method; it has the advantage of being quite spin-independent.

To be explicit, we write out the essential γ-matrix algebra contractions (following the notation of Ref. 3),

$$\bar{u}(q)(1-\gamma_5)\gamma_\alpha[2ip_\beta - i\gamma\cdot Q\gamma_\beta] \times v(p)[\mathcal{S}(k-Q)V_\beta(k-Q,k)]_{\alpha\rho}\eta_\rho, \quad (7)$$

where Q_μ = virtual photon momentum vector, V_β is the charge vertex function for the W boson and $\mathcal{S}$ is related to the W-boson propagator S by

$$S(k) = \frac{1}{i}\frac{1}{k^2+\mu^2}\mathcal{S}(k). \quad (8)$$

Ordinarily, one proceeds to carry out the four-dimensional integration over all virtual Q before contracting; clearly it would be a great help to have a rule of thumb by which we can look for and retain the log$m$ terms before the Q_μ integration. It turns out that such a rule of thumb is not difficult to arrive at; and its derivation as a lemma is presented in the Appendix. We restate the rule in a concise mathematical form, ($k^2 = -\mu^2$, $p^2 = -m^2$; $(k-p)^2 = 0$)

$$\int d^4Q(Q^2+\delta^2)^{-1}(Q^2-2p\cdot Q)^{-1}(Q^2-2K\cdot Q)^{-1} \times f(Q_\mu, Q_\nu\cdots) \xrightarrow[m\to 0]{} \frac{i\pi^2}{2}\int_0^1 d\lambda\left[\lambda + \frac{\delta}{m}\left(\frac{m}{\mu}\right)^{\frac{1}{2}}\right]^{-1} \times f(\lambda P_\mu, \lambda P_\nu\cdots)\left[4\ln\frac{\mu}{m}\right] + 0(1), \quad (9)$$

where $0(1)$ are terms which do not diverge as $m \to 0$. The essential point of this rule is that Q's everywhere in the numerator are replaced by a multiple of p. f must, for the rule to be valid, be some polynomial of Q's, p's and k's, which has an appropriate cut off in order to be definite.

The simplifications resulting from this rule of thumb, while helpful, still do not affect the spin-matrix algebra. Thus, we are still faced with an expression of the type

$$(1-\lambda)p_\beta[\mathcal{S}(k-\lambda p),\ V_\beta(k-\lambda p,\ k)]_{\alpha\rho}\eta_\rho.$$

But, by Ward's identity, we know that

$$(\lambda p)_\beta V_\beta(k-\lambda p,\ k)=-i[(k-\lambda p)^2+\mu^2]\mathcal{S}^{-1}(k-\lambda p)+i[k^2+\mu^2]\mathcal{S}^{-1}(k). \quad (10)$$

For a physical W boson, $k^2=-\mu^2$, $\mathcal{S}_{\mu\nu}{}^{-1}(k^2+\mu^2)\eta_\nu$ vanishes, and we just have

$$\frac{(1-\lambda)}{\lambda}(-i)[-2\lambda k\cdot p][\mathcal{S}(k-\lambda p)\mathcal{S}^{-1}(k-\lambda p)]_{\alpha\rho}=2ik\cdot p(1-\lambda)\delta_{\alpha\rho}. \quad (11)$$

In other words, a combination of the rule of thumb and the Ward's identity leads to the conclusion that spin plays no essential part in the vitrual photon contribution to $\log m$ terms. Precisely the same $\log m$ terms occur in the spin-zero meson. Because in the vector-meson case we do not have a $W^+\to e^+\nu\gamma$ four-point vertex, we can identify only with a nonderivative-coupling spin-zero meson interaction, viz. where the weak Lagrangian is given by $L_W\sim\phi\bar{\psi}_\nu\psi_e$. We emphasize, at this point, that the equivalence does not exist with the spin-zero vector-current interaction, even though the vector boson itself is coupled to the same vector-lepton current. The equivalence involves a switch of helicity in going from spin one to spin zero.

Having thus far shown that for the virtual photons the contribution of $\log m$ terms to the ratio of rates is equivalent to the spin-zero scalar current case, we can proceed to quote the total contribution from Fig. 2 as

$$\Delta R(\text{virtual photon})=R_0\frac{\alpha}{\pi}\left[2\ln\frac{m_W}{m_e}\ln\frac{\delta}{m_e}-2\ln\frac{m_W}{m_\mu}\ln\frac{\delta}{m_\mu}-\left(\ln\frac{m_W}{m_e}\right)^2+\left(\ln\frac{m_W}{m_\mu}\right)^2+\tfrac{1}{2}\ln\frac{m_\mu}{m_e}\right], \quad (12)$$

where δ = fictitious photon mass.

The explicit dependence of ΔR(virtual) on the photon mass is, of course, to be cancelled by a proper inclusion of the bremsstrahlung photon effect. To complete the proof of equivalence of vector meson with a spin-zero meson, it is necessary to show that it is valid for real photons as well. This we shall show in the next section.

II. $W^+\to l^++\nu+\gamma$

There are two diagrams which contribute to the inner bremsstrahlung matrix element (see Fig. 3). The

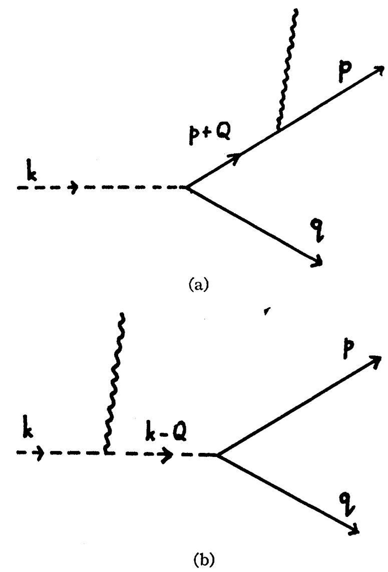

FIG. 3. Perturbation diagrams for lowest order real photon correction to the $W^+\to e^+\nu$ process.

matrix element is, explicitly,

$$eg_0\bar{u}(q)(1-\gamma_5)\left\{\gamma_\rho\eta_\rho\left[\frac{p\cdot\epsilon}{p\cdot Q}+\frac{\gamma\cdot Q\gamma\cdot\epsilon}{2p\cdot Q}\right]+\gamma_\alpha[S(k-Q)V_\beta(k-Q,\ k)]_{\alpha\rho}\eta_\rho\epsilon_\beta\right\}v(p). \quad (13)$$

The first part of the matrix element comes from the graph where the electron throws off a real photon; the second part comes from the vector boson throwing off a real photon.

The actual method of treatment of the real photon effects used here is the customary one of a noncovariant integration in the decaying meson rest frame, employing

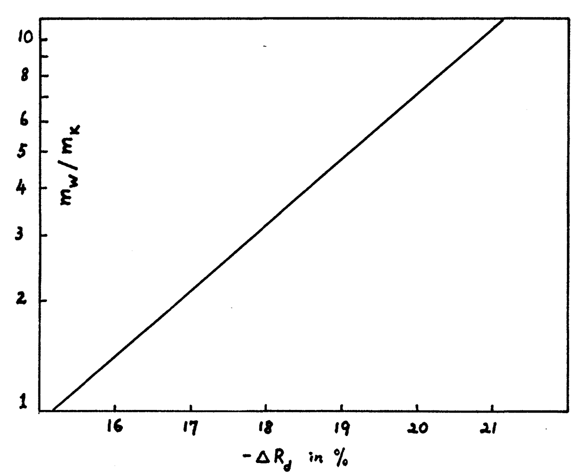

FIG. 4. A plot of the dependence of radiative correction to R_0 with photon discrimination, $\Delta R_{d'}$ on the $\ln(m_W/m_K)$. The photon discrimination has been taken to be $2m_e$ in both decay modes.

a Coester's representation[5] for the photons. With this method, the log$m$ terms arise entirely from the electron graph, Fig. 3(a), as we now show.

Let us first note the origin of log$m$ terms within the framework of the noncovariant integration outlined above. It clearly comes from the configuration where the electron and the photon come out parallel to each other. Specifically, the term $(p\cdot\epsilon)/(p\cdot Q)$ behaves as $(m=0)$.

$$\frac{-\sin\theta}{(\cos\theta-1)Q}\xrightarrow{\theta\to 0}\frac{2}{\theta}\frac{1}{Q},$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{Q}$. This divergence, when squared and weighted over angles $\theta d\theta$, clearly leads to the logarithmic divergence we seek.

A more general characterization of the origin of log$m$ terms can be obtained by means of a $m=0$ representation for $v(p)$ where

$$v(p)=\begin{pmatrix}0\\ i\sigma_2\chi_\downarrow\end{pmatrix}.$$

The z axis has been chosen along the $\mathbf{p}$ direction. The σ_2 serves to flip the helicity so that the positron is right-handed. In this representation, then,

$$Jv(p)=\left(\frac{p\cdot\epsilon}{p\cdot Q}+\frac{\gamma\cdot Q\gamma\cdot\epsilon}{2p\cdot Q}\right)v(p)=\left(\frac{p\cdot\epsilon}{p\cdot Q}+\frac{1}{2p\cdot Q}\right.$$
$$\left.\times[i\hat{p}\cdot(\mathbf{Q}\times\boldsymbol{\varepsilon})-i\hat{p}\cdot\boldsymbol{\varepsilon}Q_4]\right)\begin{pmatrix}0\\ i\sigma_2\chi_\downarrow\end{pmatrix}+A\begin{pmatrix}0\\ \chi_\downarrow\end{pmatrix},\quad(14)$$

where A is some unit matrix in spinor space depending on $\mathbf{Q}$, $\mathbf{p}$ and $\boldsymbol{\varepsilon}$, which we do not care to write out. We might just note that A behaves at $\theta\to 0$ as $1/\theta^2$.

A, however, cannot contribute because of the $(1+\gamma_5)$ operator which acts on $Jv(p)$. In other words, we can immediately replace J by a constant in spinor space, so that the electron bremsstrahlung effect is just a multiple of the bare-matrix element.

It is easy to see why the boson bremsstrahlung does not contribute to log$m$ terms. For at $\theta=0$, the boson terms do not diverge, and with the A terms absent the cross terms in the square of the matrix element can diverge at most as $1/\theta$. Upon weighting with $\theta d\theta$, this divergence vanishes.

The identification of the spin-zero analog is, by now, straightforward. For in the spin-zero scalar-current case, the matrix element is

$$\bar{u}(q)(1-\gamma_5)\left[\frac{p\cdot\epsilon}{p\cdot Q}+\frac{\gamma\cdot Q\gamma\cdot\epsilon}{2p\cdot Q}\right]v(p).\qquad(15)$$

Again only the correct helicity term in $Jv(p)$ contributes.

The log$m$ terms coming from a spin-zero scalar interaction case is readily computed from the matrix element. It has also been computed by Kinoshita in his work on the $\pi\to e\nu$ problem.[4] We quote the result for the ratio

$$\Delta R(\text{real photon})$$
$$=R_0\left\{\frac{\alpha}{\pi}\left[-2\ln\frac{m_W}{m_e}\ln\frac{\delta}{m_W}+2\ln\frac{m_W}{m_\mu}\ln\frac{\delta}{m_W}\right.\right.$$
$$\left.\left.-\left(\ln\frac{m_W}{m_e}\right)^2+\left(\ln\frac{m_W}{m_\mu}\right)^2-\tfrac{1}{2}\ln\frac{m_\mu}{m_e}\right]\right\}.\quad(16)$$

Combining the two results (16) and (12), we see that the radiative correction to the ratio of total rates is zero in our approximation. As we have been careful to point out all along, this result is closely related to the vanishing of the log$m$ terms in the spin-zero scalar-current case. In other words, the spin of the vector meson was almost entirely unessential as far as the log$m$ terms are concerned. In fact, as we shall show in the next section, this spin independence is true for all spins.

III. SPIN INDEPENDENCE FOR ARBITRARY BOSON SPINS

For clarity, we shall study first the case of a meson of arbitrary spin whose weak-interaction Lagrangian is of the form

$$W_{\rho\cdots\sigma}\bar{\psi}_\nu(1-\gamma_5)O_{\rho\cdots\sigma}\psi_l.\qquad(17)$$

Keeping in mind the subsidiary conditions on $W_{\rho\cdots\sigma}$, viz.

$$\begin{aligned}&\text{(i)}\quad W_{\rho\cdots\sigma}\text{ is completely symmetric},\\&\text{(ii)}\quad W_{\cdots\mu\nu\cdots}\delta_{\mu\nu}=0\text{ (traceless tensor)},\\&\text{(iii)}\quad \partial_\mu W_{\cdots\mu\cdots}=0,\end{aligned}\qquad(18)$$

we see that $O_{\rho\cdots\sigma}$ can only be of the form

$$\text{(i)}\quad \partial_\rho\cdots\partial_\sigma,$$

or

$$\text{(ii)}\quad \gamma_\rho\partial_\tau\cdots\partial_\sigma.$$

where the ∂'s are chosen to operate on ψ_ν only. The actual form of O is not important, except that it must satisfy further the physically reasonable condition that O cannot generate both helicity states of the lepton (at $m=0$). This means that we do not allow an admixture of the two forms of O mentioned above. An alternate statement of this condition would be that O must satisfy

$$\gamma_5 O_{\rho\cdots\sigma}=\pm O_{\rho\cdots\sigma}\gamma_5.\qquad(19)$$

The arguments for a spin-zero analog for the virtual photon graphs go through much as before in the vector meson case. The Ward's-identity argument holds whatever the spin. Thus, we would find for the graph where the photon is emitted by the meson and absorbed by the electron a term

$$2ik\cdot p(1-\lambda)\bar{u}(q)(1-\gamma_5)O_{\rho\cdots\sigma}v(p)\eta_{\rho\cdots\sigma}.$$

The spin-zero analog of such a term is obvious.

[5] J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955).

For the real photon case, condition (19) is important. Because of it, the wrong helicity term in $Jv(p)$ never contributes and consequently the meson-bremsstrahlung part of the matrix element also does not contribute. Since $Jv(p)$ becomes a multiple of the bare-matrix element the spin is again trivial.

In other words, we have shown that as far as log$m$ terms are concerned an arbitrary spin-case Lagrangian (17) is entirely equivalent to the spin-zero Lagrangian.

For a more general Lagrangian of the form

$$[(\partial_\mu\ldots)(\partial_\tau\ldots)W_{\rho\ldots\sigma}](\partial_\mu\ldots\bar{\psi}_\nu)O_{\rho\ldots\sigma}(\partial_\tau\ldots\psi_l)\,, \quad (20)$$

the spin-zero analog is just

$$[(\partial_\mu\ldots)(\partial_\tau\ldots)\phi](\partial_\mu\ldots\bar{\psi}_\nu)(\partial_\tau\ldots\psi_l)\,. \quad (21)$$

In this section we have shown that, to lowest order in α, the spin of the meson is entirely immaterial in the log$m$ terms. We have, of course, not shown this to be true for all orders in α.

The class of theories included in (21) may, in actual fact, not be all physically reasonable. Thus, we can find many such examples of interaction which in turn lead to nonvanishing log$m$ terms in the total ratio of rates, to lowest order in α. One such example would be a weak decay of a spin 0 meson described by a Lagrangian

$$(\partial_\mu\partial_\rho\phi)\bar{\psi}_l\partial_\mu\partial_\rho\psi_\nu\,.$$

The result for the ratio of rates is

$$R=R_0\{1+\alpha/\pi\tfrac{1}{6}\ln m_\mu/m_e\}\,.$$

This theory may be unreasonable because it does not belong in the Hamiltonian framework in a simple way.

IV. RATIO R WITH DISCRIMINATION

In conclusion, we are including a section on a measurement of the ratio of partial rates. That is, we are indicating here the result for the W-boson case where discrimination against events with bremsstrahlung photons of energy greater than Q_1 is applied. The practical aspects of such an experimental setup clearly belong to the remote future; we quote the result because, being nonvanishing in our approximation, it is numerically more definitive than the ratio of total rates.

For the discrimination case, it is very convenient to limit outselves to the region where Q_1 the descrimination energy, is small compared with the boson mass. This means that Q_1 need not be extremely small by itself for the result we quote below to be valid. This also means theoretically that only the $(p\cdot\epsilon)/(p\cdot Q)$ part of the bremsstrahlung-matrix element contributes. The spin independence of that part of the matrix element is obvious. We can therefore carry over the corresponding result that has already been obtained for the K-meson case.[6]

[6] N. P. Chang, Phys. Rev. **129**, 399 (1963).

The final result is $(Q_1/m_W\ll 1)$

R(discrimination)

$$=R_0\left\{1-\frac{2\alpha}{\pi}\left[-\ln\frac{Q_1{}^\mu}{Q_1{}^e}+\ln\frac{m_W}{m_e}\ln\frac{m_W}{2Q_1{}^e} -\ln\frac{m_W}{m_\mu}\ln\frac{m_\mu}{2Q_1{}^\mu}+\left(\ln\frac{m_W}{m_e}\right)^2-\left(\ln\frac{m_W}{m_\mu}\right)^2 -\tfrac{3}{4}\ln\frac{m_\mu}{m_e}+0\left(\frac{m_e{}^2}{m_W{}^2},\frac{m_\mu{}^2}{m_W{}^2}\right)\right]\right\}. \quad (24)$$

Except for the last term, Eq. (24) is a transcript (to order $m^2/m_W{}^2$) of the corresponding $K\to(e\nu)/(\mu\nu')$ ratio of Ref. 6. The last term is different because of the absence of $W\to l\nu\gamma$ vertex which is present in the K-derivative coupling case.

Numerically, the radiative correction, much like the K-meson case, is quite large; the actual magnitude depends, of course, on the value of m_W and the experimental discrimination energy. When $m_W=m_k$, the correction is almost 15%. For orientation purposes, we have indicated in a plot the dependence of R_d on m_W.

ACKNOWLEDGMENTS

We wish to thank Professor T. D. Lee for suggesting this investigation and for an enlightening discussion. We also wish to thank Professor T. Kinoshita and Professor G. Feinberg for useful comments.

APPENDIX

In this Appendix we shall study a little more closely the log$m$ terms in a Feynman integral. For simplicity, we begin with a study of the general class of integrals

$$\int d^4Q\frac{Q_\sigma Q_\tau\ldots Q_\nu}{(Q^2)(Q^2-2p\cdot Q)}\left[\frac{\Delta^2}{Q^2+\Delta^2}\right]^{n+1}=J_{\sigma\tau\ldots\nu}(p)\,, \quad (\mathrm{A1})$$

where $n=$ number of Q's in the numerator. The convergence factor is introduced to give meaning to the integrals. Now we shall prove the following lemma:

Lemma 1: Logarithmic divergence in m occurs only in that part of the tensor $J_{\sigma\tau\ldots\nu}(p)$ which transforms as $p_\sigma p_\tau\cdots p_\nu$.

Proof: This involves looking explicitly at the generating function, $I(q)$, defined by

$$I(q)\equiv\int d^4Q\frac{1}{Q^2}\frac{1}{Q^2-2p\cdot Q'}\cdot\frac{\Delta^2}{Q^2-2q\cdot Q+\Delta^2} \quad (\mathrm{A2})$$

$$=\frac{i\pi^2}{2}\int_0^1 dx\int_0^1 dy2(1-y) \times\sum_{n=0}^{\infty}\frac{[2xy(1-y)p\cdot q+y^2q^2]^n}{[\Delta^2y+x^2(1-y)^2m^2]^{n+1}}\Delta^2. \quad (\mathrm{A3})$$

Clearly, by inspection, log$m$ divergence occurs in that part of the series involving $(p\cdot q)$ only. By means of the relation

$$J_{\sigma\tau\cdots\nu}(p)=\left(\frac{\Delta^2}{2}\right)^n\frac{1}{n!}\left[\frac{\partial}{\partial q_\sigma}\frac{\partial\ldots}{\partial q_\tau}\frac{\partial}{\partial q_\nu}I(q)\right]_{q=0} \tag{A4}$$

we come immediately to the conclusion stated above in the lemma.

As a matter of fact, we can evaluate the coefficient of $\log m$ term in general, using the integral representation (A3). Thus we find

$$I(q)=\frac{i\pi^2}{2}\sum_{n=0}^{\infty}\left(\frac{2}{\Delta^2}\right)^{n+1}\frac{(p\cdot q)^n}{(n+1)}\left[\ln\frac{\Delta^2}{m^2}\right]\Delta^2+O(1)$$

where $O(1)$ are terms which remain finite as $m\to 0$. The corresponding expression for $J_{\sigma\tau\cdots\nu}(p)$ becomes

$$J_{\sigma\tau\cdots\nu}(p)=p_\sigma p_\tau\cdots p_\nu\frac{i\pi^2}{2}\left[\frac{4}{n+1}\ln\frac{\mu}{m}\right]+O(1)\,, \tag{A5}$$

where n, we recall, is the rank of the tensor $J_{\sigma\tau\cdots\nu}$. In (A5) we have used μ instead of Δ as the scale for the logarithm; this change of scale is, of course, immaterial in the final ratio of rates.

The generalization to the class of integrals we actually need is straightforward. Let

$$\int d^4Q\frac{Q_\sigma Q_\tau\cdots Q_\nu Q_\lambda}{(Q^2)(Q^2-2p\cdot Q)}\frac{1}{(Q^2-2k\cdot Q)}\left[\frac{\Delta^2}{Q^2+\Delta^2}\right]^{n+1}=K_{\sigma\tau\cdots\nu\lambda}(p,k)\,. \tag{A6}$$

Then a similar lemma holds true, viz.

Lemma 2: Logarithmic divergence in m occurs only in that part of the tensor $K_{\sigma\tau\cdots\nu\lambda}$ which transforms as $p_\sigma p_\tau\ldots p_\nu p_\lambda$.

Proof: We note that, as far as $\log m$ terms are concerned, for which it is the $Q\to 0$ part of integration that is important, the contraction relation holds

$$(-2k_\lambda)K_{\sigma\tau\cdots\nu\lambda}=J_{\sigma\tau\cdots\nu}+O(1) \tag{A7}$$

$O(1)$ being terms which remain finite as $m\to 0$. Thus we conclude that there are only two places where $\log m$ can appear in K tensor, i.e.

$$a(p_\sigma p_\tau\cdots p_\nu)k_\lambda+b(p_\sigma p_\tau\cdots p_\nu)p_\lambda\,.$$

But $p_\lambda K_{\sigma\tau\cdots\nu\lambda}$ certainly does not have $\log m$ at all, so that a cannot have $\log m$. Hence, the lemma follows.

In fact, from (A5) we conclude that (for $p\cdot k=-\frac{1}{2}\mu^2+m^2$)

$$K_{\sigma\tau\cdots\nu\lambda}=p_\sigma p_\tau\cdots p_\nu p_\lambda\frac{i\pi^2}{2}\left[\frac{4}{n'}\frac{1}{\mu^2}\ln\frac{\mu}{m}\right]+O(1)\,, \tag{A8}$$

where n' = rank of the tensor $K_{\sigma\tau\cdots\nu\lambda}$. The result (A8) holds for all $n'\geq 1$, $n'=0$ has infrared divergence. A separate integration for $n'=0$ shows

$$K=\frac{i\pi^2}{2}\left[\frac{2}{\mu^2}\ln\frac{\mu}{m}\left(\ln\frac{\mu}{m}-2\ln\frac{\delta}{m}\right)\right]+O(1)\,. \tag{A9}$$